1. Linear groups - some basic facts

We discuss a few things about subgroups of $GL(n, E)$ where $E = \mathbb{R}$ or $\mathbb{C}$. These linear groups are examples of non-abelian topological groups. We discuss them separately here since most of our later discussions will be about abelian topological groups. Think of $GL(n, E)$ as a subspace of $E^{n^2}$.

[130] The space $GL(n, E)$ is open and dense in $E^{n^2}$. The space $GL(n, \mathbb{C})$ is path connected and hence connected; but $GL(n, \mathbb{R})$ is not connected.

**Proof.** $GL(n, E)$ is open since $GL(n, E) = \det^{-1}(E \setminus \{0\})$ and determinant is a continuous function (being a finite sum of products of projections) [as an aside it may be noted that $\det : GL(n, E) \to E \setminus \{0\}$ is also a group homomorphism]. Denseness is clear since given any $n \times n$ matrix over $E$ and $\epsilon > 0$, we can modify the entries of the matrix upto $\pm \epsilon$ to obtain a matrix with non-zero determinant.

Let $G = GL(n, \mathbb{C})$. If $A \in G$ is an upper triangular matrix, it is easy to see that there is a path in $G$ connecting $A$ and the identity matrix $I$ (work with each entry separately). Now if $B \in G$, then there is $P \in G$ such that $PBP^{-1}$ is upper triangular [p.130, Artin’s Algebra]. Conjugation by $P$ is a homeomorphism of $G$ so that if $\alpha$ is a path in $G$ connecting $I$ to $PBP^{-1}$, then $P^{-1}\alpha P$ is a path in $G$ connecting $I$ and $B$. Thus $G = GL(n, \mathbb{C})$ is path connected. But $GL(n, \mathbb{R})$ is not connected since $det : GL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is onto (check) and $\mathbb{R} \setminus \{0\}$ is not connected. 

**Remark:** It can be shown (using the rational canonical form) that $GL(n, \mathbb{R})$ has exactly two (path) connected components, $\{A : det[A] > 0\}$ and $\{A : det[A] < 0\}$. Try to find a path in $GL(2, \mathbb{R})$ from $I$ to $-I$ to get a feeling of the problem.

Some standard subgroups of $GL(n, E)$ are:

$SL(n, E) = \ker(det) = \{A \in GL(n, E) : det[A] = 1\}$ [linear maps preserving volume and orientation],
Theorem

\[ \sum \text{these are closed conditions}, \]

\[ \text{fixing} \]

\[ \sum \]

\[ \text{Exercise-18} \]

\[ \square \]

\[ \text{rotation by angle} \]

\[ \text{Proof.} \]

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Proof. Let $A \in SO(3)$. Then there is a unit vector $v_1 \in \mathbb{R}^3$ such that $Av_1 = v_1$ by Exercise-18. Extend to an orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$. Let $P \in O(3)$ be the matrix with columns $v_1, v_2, v_3$. Then $P^{-1}AP \in SO(3)$ and $P^{-1}AP$ has the form

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & B & 0 \\
0 & 0 & B
\end{pmatrix},
\]

where $B \in SO(2)$ necessarily. Since $B$ is a rotation, we have that $A$ is a rotation in the plane spanned by $\{v_2, v_3\}$ around the axis given by $v_1$. Conversely, given a unit vector $v \in \mathbb{R}^3$ and angle $\theta$, the rotation by angle $\theta$ around the axis given by $v$ and fixing $v$, is easily seen to be a member of $SO(3)$ ($\therefore$ after conjugating by an element of $O(3)$, assume $v = e_3$).

An argument for showing that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$ is possibly the following [see Artin’s Algebra for the details]. Consider $\{(v, \theta) : v \in S^2, 0 \leq \theta < 2\pi\}$, which may be identified with $S^3$. Now, the antipodal points $(v, \theta)$ and $(-v, -\theta) = (-v, 2\pi - \theta)$ represent the same rotation. If we identify $(v, \theta)$ and $(-v, -\theta)$, we get the real projective space $\mathbb{R}P^3$. \hfill $\square$

[134] $SU(2)$ is homeomorphic to $S^3$ (in particular, it follows that $S^3$ admits a group structure compatible with the topology).

Proof. Let $A = \begin{pmatrix} a & b \\ d & c \end{pmatrix} \in SU(2)$, $a, b, c, d \in \mathbb{C}$. Since $A^{-1} = A^*$, we have $c = -\overline{b}$ and $d = \overline{a}$ so that $A = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$. Now using $\det[A] = 1$, we have $|a|^2 + |b|^2 = 1$. If $a = x_1 + ix_2$, $b = x_3 + ix_4$, then $A$ corresponds to $(x_1, x_2, x_3, x_4) \in S^3$. \hfill $\square$

Remark: It is a fact that $S^n$ admits a group structure compatible with the topology iff $n = 1$ or 3.

Latitudes and longitudes on $S^3$ have algebraic interpretations in terms of the conjugacy classes in $SU(2)$ [p.274, Artin’s Algebra]. Fix $c \in (-1, 1)$. Then the corresponding latitude is $\{(x_1, x_2, x_3, x_4) \in S^3 : x_1 = c\}$, which is a 2-sphere of radius $\sqrt{1-c^2}$. This latitude is precisely $\{A \in SU(2) : \text{trace}[A] = 2c\}$ (\because if $A = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ with $a = x_1 + ix_2$ and $b = x_3 + ix_4$, then $\text{trace}[A] = 2x_1$), and the conjugacy classes in $SU(2)$ are these latitudes together with $\{I\}, \{-I\}$ (corresponding to $c = \pm 1$).

Longitudes of $S^3$ are the circles obtained by intersecting $S^3$ with planes containing the north and south poles. If $P = \{x \in S^3 : x_3 = x_4 = 0\}$, then the longitude $S^3 \cap P$ is the collection of all diagonal matrices in $SU(2)$, which is a subgroup. All other longitudes are conjugates of this subgroup.

Let $G = \{A \in SU(2) : \text{trace}[A] = 0\}$, which corresponds to the equator on $S^3$. Every $B \in SU(2)$ acts on $G$ by conjugation: $f_B : G \to G$, $f_B(A) = BAB^{-1}$. On the 2-sphere $G$, $f_B$ acts as a rotation and then $f_B$ can be thought of as a rotation of $\mathbb{R}^3$ and thus $f_B \in SO(3)$. [If $A \in G$, then it can be seen that $A$ is skew-Hermitian, i.e., $A^* = -A$; the collection of all $2 \times 2$ skew-Hermitian matrices has real dimension 3, and this collection plays the role of $\mathbb{R}^3$ for $f_B$ to act, see p.278, Artin.] The

\begin{align*}
\text{Latitudes} & : (x_1, x_2, x_3, x_4) \in S^3 \quad \text{with} \\
\text{Longitudes} & : x_1 = \text{constant}.
\end{align*}
map $B \mapsto f_B$ from $SU(2)$ to $SO(3)$ is a surjective group homomorphism with kernel $\{\pm I\}$. Hence people say that:

[135] $SU(2)$ is a double cover of $SO(3)$.

$SL(2,\mathbb{R}$ and $SL(2,\mathbb{C})$: If $v \in \mathbb{R}^2 \setminus \{0\}$, let $\Gamma_v = \{rv : r \geq 0\}$ be the ray determined by $v$. If $A \in SL(2,\mathbb{R})$, then $A(\Gamma_v) = \Gamma_{A(v)}$. Let $G = \{A \in SL(2,\mathbb{R}) : A(\Gamma_{e_1}) = \Gamma_{e_1}\}$. Then $G$ consists of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$, where $a > 0$ and $b \in \mathbb{R}$. If $A \in SL(2,\mathbb{R})$, then there is a unique rotation $B \in SO(2)$ such that $B(\Gamma_{e_1}) = \Gamma_{A(e_1)} = A(\Gamma_{e_1})$. Then $C := B^{-1}A \in G$, and this $C$ is unique since $B$ was unique. This says that the map $(B,C) \mapsto BC$ from $SO(2) \times G$ to $SL(2,\mathbb{R})$ is a bijection. It may be verified that this map is continuous both ways. Also note that $G$ is homeomorphic to $(0,\infty) \times \mathbb{R} \cong \mathbb{R}^2$. Hence we have:

[136] The map $f : SO(2) \times G \rightarrow SL(2,\mathbb{R})$, $f((B,C)) = BC$ is a homeomorphism (not a group homomorphism). Consequently, $SL(2,\mathbb{R})$ is homeomorphic to $S^1 \times \mathbb{R}^2$.

**Exercise-19:** Let $f : C \rightarrow C$ be an injective entire function. Then $f(z) = az + b$ for some $a \in C \setminus \{0\}$ and $b \in C$. [Hint: Enough to show $f$ is a polynomial, for then $|f^{-1}(w)| = deg(f)$ except for finitely many $w \in \mathbb{C}$ (: $f'$ vanishes if there is a repeated solution). Since $z \mapsto f(1/z)$ is one-one, 0 cannot be an essential singularity of $f(1/z)$ by Casorati-Weierstrass Theorem and Open Mapping Theorem. By considering the power series expansion, $f$ is a polynomial.]

**Exercise-20:** Let $f(z) = \frac{az + b}{cz + d}$ be a Mobius map such that $f(\mathbb{R} \cup \{\infty\}) \subset \mathbb{R} \cup \{\infty\}$. Then $a,b,c,d \in \mathbb{R}$. [Hint: There is a Mobius map $g$ with real coefficients such that $g$ takes the three points $f(0), f(1), f(\infty) \in \mathbb{R} \cup \{\infty\}$ to $0,1,\infty$ respectively. Then $g \circ f$ fixes $0,1,\infty$ and hence $g \circ f = Id$ or $f = g^{-1}$.

**Exercise-21:** Let $D$ be the open unit disc in $\mathbb{C}$. If $f : D \rightarrow D$ is a biholomorphism fixing 0, then $f$ is a rotation. [Hint: By Schwarz Lemma, $|f(z)| \leq |z|$ and $|f^{-1}(w)| \leq |w|$ for every $z,w \in D$. Taking $w = f(z)$ we have $|z| = |f^{-1}(f(z))| \leq |f(z)|$ so that $|f(z)| = |z|$. Apply Schwarz once more.]

$SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ are naturally related to the theory of Riemann surfaces. Let $Aut(\mathbb{C}_\infty)$ and $Aut(\mathbb{H})$ be the groups of all biholomorphisms of the Riemann sphere $\mathbb{C}_\infty$ and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ respectively.

[137] We have the following group isomorphisms: $Aut(\mathbb{C}_\infty) = SL(2,\mathbb{C})/\{\pm I\}$ and $Aut(\mathbb{H}) = SL(2,\mathbb{R})/\{\pm I\}$.

**Proof.** Let $\mathcal{M}$ be the group of all Mobius maps. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, then we have the corresponding Mobius map $f_A(z) = \frac{az + b}{cz + d}$ which belongs to $Aut(\mathbb{C}_\infty)$. Moreover, $A$ and $-A$ induce the same Mobius map, and so $A \mapsto f_A$ from $SL(2,\mathbb{C})$ to $\mathcal{M}$ is a surjective group
homomorphism with kernel \( \{ \pm I \} \). Thus it suffices to show that every \( f \in Aut(\mathbb{C}_\infty) \) is a Mobius map.

After composing with a Mobius map, assume \( f(0) = 0 \) and \( f(\infty) = \infty \). Then \( f|_C \) is an entire function. There are two arguments to say \( f \) is a Mobius map.

**Argument-1:** Since \( \lim_{z \to 0} f(z)/z = f'(0) \in \mathbb{C} \), \( f(z)/z \) is bounded in a deleted neighborhood of 0. Let \( \phi(z) = 1/z \). Then \( \phi \in \mathcal{M} \) and \( \phi^{-1} = \phi \). If \( g(z) = (\phi \circ f \circ \phi)(z) = 1/f(1/z) \), then \( g \) fixes 0 and \( \infty \), so that \( g|_C \) is entire. Since \( g \) has an inverse \( \phi \circ f^{-1} \circ \phi \), \( g' \) never vanishes on \( \mathbb{C} \). In particular, \( \lim_{z \to \infty} f(z)/z = \lim_{w \to 0} w/g(w) = 1/g'(0) \in \mathbb{C} \). So \( f(z)/z \) is bounded in \( \mathbb{C} \backslash K \) for some compact \( K \subset \mathbb{C} \). Thus \( f(z)/z \) is bounded on \( \mathbb{C} \backslash \{0\} \). Hence the entire function \( z \mapsto f(e^z)/e^z \) is constant, say \( c \neq 0 \), by Liouville's Theorem. So \( f(z) = cz \), which is a Mobius map.

**Argument-2:** Since \( f|_C \) is an injective entire function, \( f(z) = az + b \) by Exercise-19 (in fact, \( b = 0 \) since \( f(0) = 0 \)).

Let \( \mathcal{M}_\mathbb{R} \) be the subgroup of all Mobius maps with real coefficients. It may be seen that \( SL(2, \mathbb{R})/\{ \pm I \} \) is isomorphic to \( \mathcal{M}_\mathbb{R} \). So it suffices to show \( \mathcal{M}_\mathbb{R} = Aut(\mathbb{H}) \). Consider \( f \in \mathcal{M}_\mathbb{R} \), \( f(z) = \frac{az + b}{cz + d} \), where we may assume \( ad - bc = 1 \). Then \( f(\mathbb{R} \cup \{ \infty \}) \subset \mathbb{R} \cup \{ \infty \} \) and \( \text{Im}(f(i)) = 1/(c^2 + d^2) > 0 \) so that \( f(i) \in \mathbb{H} \). We conclude that \( f \in Aut(\mathbb{H}) \). Conversely, consider \( f \in Aut(\mathbb{H}) \). Since \( \mathcal{M}_\mathbb{R} \) acts transitively on \( \mathbb{H} \) (\( z \mapsto az + b \) takes \( i \) to \( b + ia \)), we may assume \( f(i) = i \). Now there is \( \phi \in \mathcal{M} \) (in fact, \( \phi(z) = \frac{i - z}{i + z} \)) taking \( \mathbb{H} \) biholomorphically onto the open unit disc \( D \) in such a way that \( \phi(i) = 0 \). Then \( \phi \circ f \circ \phi^{-1} \) is a biholomorphism of \( D \) fixing 0 and therefore it is a rotation, say \( g \), by Exercise-21. Then, \( f = \phi^{-1} \circ g \circ \phi \in \mathcal{M} \) and also \( f(\mathbb{R} \cup \{ \infty \}) \subset \mathbb{R} \cup \{ \infty \} \).

Hence by Exercise-20, \( f \in \mathcal{M}_\mathbb{R} \).

**Remarks:** (i) Any Mobius map can be written as a finite composition of: translations \( (z \mapsto z + b, \ b \in \mathbb{C}) \), dilations \( (z \mapsto az, \ a \in \mathbb{C} \backslash \{0\}) \), and -(inversion) \( (z \mapsto -1/z) \). The corresponding matrices in \( SL(2, \mathbb{C}) \) are \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \) \((b \in \mathbb{C})\), \( \begin{pmatrix} b & 1 \\ 0 & 1/b \end{pmatrix} \) \((b \in \mathbb{C} \backslash \{0\}, \ b^2 = a)\), \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). They generate \( SL(2, \mathbb{C})/\{ \pm I \} \) and hence \( SL(2, \mathbb{C}) \) since \( -I \) is of the form \( \begin{pmatrix} b & 1 \\ 0 & 1/b \end{pmatrix} \).

(ii) Suppose \( f \in Aut(\mathbb{H}) \). If \( f(\infty) \neq \infty \), then compose with \( z \mapsto z - f(\infty) \) and \( z \mapsto -1/z \) and assume \( f(\infty) = \infty \). If \( f(z) = \frac{az + b}{cz + d} \), then \( c = 0 \) so that \( f(z) = (az/d) + b/d \). Since \( 1 = \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = ad \), we have \( a/d > 0 \). Also \( b/d \in \mathbb{R} \) since \( d \neq 0 \). This argument shows that \( SL(2, \mathbb{R})/\{ \pm I \} \) is generated by the following types of matrices: \( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \) \((y \in \mathbb{R})\), \( \begin{pmatrix} y & 1 \\ 0 & 1/y \end{pmatrix} \).
(y > 0), \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \): We may write \( y = e^t \) so that \( 1/y = e^{-t} \). Also note that \(-I\) and \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) can be written in the form \( \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \). Summarizing:

\[ \text{[138]} \]

(i) \( SL(2, \mathbb{C}) \) is generated by: \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} (b \in \mathbb{C}) \), \( \begin{pmatrix} b & 1 \\ 0 & 1/b \end{pmatrix} (b \in \mathbb{C} \setminus \{0\}) \), and \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

(ii) \( SL(2, \mathbb{R}) \) is generated by: \( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} (t \in \mathbb{R}) \), \( \begin{pmatrix} e^t & 1 \\ 0 & e^{-t} \end{pmatrix} (t > 0) \), and \( \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \) \((t \in \mathbb{R})\).

An informal introduction to the Lie algebra of some linear groups: Consider \( GL(n, \mathbb{R}) \), \( SL(n, \mathbb{R}) \), \( O(n) \) for our discussion (complex analogues can be treated similarly). They are examples of Lie groups (i.e., smooth manifolds having compatible group structure). If \( G \) is a Lie group with identity \( e \), then the tangent space at \( e \) to \( G \) is called the Lie algebra of \( G \). So the Lie algebra is a vector space (it has additional structures). Some advantages in working with the Lie algebra are: (i) \( G \) may not be commutative but the addition in the Lie algebra is commutative, (ii) \( G \) may have nontrivial ‘curvature’, but the Lie algebra is ‘flat’.

We loosely define a tangent vector: \( v \in T_eG \) iff there is a smooth path \( \alpha \) on \( G \) such that \( \alpha(0) = e \) and \( \alpha'(0) = v \). Now by Taylor expansion, such \( \alpha \) can be written as \( \alpha(t) = e+tv+E(t^2) \), where \( E(t^2) \) denotes a negligible error term containing powers \( t^k \) with \( t \geq 2 \). Hence if \( G = GL(n, \mathbb{R}), SL(n, \mathbb{R}) \) or \( O(n) \), then \( T_I(G) = \{ M \in M(n, \mathbb{R}) : I+tM \in G \text{ up to an error of } t^2 \text{ for small } t \} \). We have:

\[ \text{[139]} \]

(i) If \( G = GL(n, \mathbb{R}) \), then \( T_I G = M(n, \mathbb{R}) \).

(ii) If \( G = SL(n, \mathbb{R}) \), then \( T_I G = \{ A \in M(n, \mathbb{R}) : \text{trace}[A] = 0 \} \).

(iii) If \( G = O(n) \), then \( T_I G = \{ A \in M(n, \mathbb{R}) : A^t = -A \} \) (skew-symmetric matrices).

**Proof.** (i) It is intuitively clear since \( GL(n, \mathbb{R}) \) is open in \( M(n, \mathbb{R}) \). Or, note that for any \( M \in M(n, \mathbb{R}) \), \( \det[I + tM] \approx \det[I] = 1 \) so that \( \det[I + tM] \neq 0 \) for small \( t \).

(ii) \( T_I G = \{ A \in M(n, \mathbb{R}) : \det[I + tM] \neq 0 \text{ up to an error of } t^2 \text{ for small } t \} \). If \( M = [m_{ij}] \), then

\[
\det[I + tM] = \prod_{i=1}^n (1 + tm_{ii}) + E_1(t^2) + E_2(t^2) + E_3(t^2) = 1 + t(\sum_{i=1}^n m_{ii}) + E(t^2) = 1 + t(\text{trace}[M]) + E(t^2) = 1 + t\text{trace}[M] + E(t^2).
\]

Hence \( \det[I + tM] = 1 \) up to an error of \( t^2 \) for small \( t \) iff \( \text{trace}[M] = 0 \).

(iii) \( T_I G = \{ A \in M(n, \mathbb{R}) : [I + xM]^t = [I + xM]^{-1} \text{ up to an error of } x^2 \text{ for small } x \} \). But \( [I + xM][I + xM]^t = [I + xM][I + xM]^t = I + x(M + M^t) + x^2MM^T \). This is equal to \( I \) up to an error of \( x^2 \) iff \( M + M^t = 0 \).

If \( G = U(n) \), then we can show \( T_I G = \{ A \in M(n, \mathbb{C}) : A^* = -A \} \).

If we think of the ‘dimension’ of \( G \) as that of \( T_I G \), then we may deduce:
Real dimension of some linear groups are as follows: $\dim[SL(n, \mathbb{R})] = n^2 - 1$, $\dim[SL(n, \mathbb{C})] = 2n^2 - 2$, $\dim[O(n)] = n(n - 1)/2$, and $\dim[U(n)] = n(n - 1)$.

2. Topological groups - basics

We say $(X, T, \ast)$ is a topological group if (i) $(X, T)$ is a $T_1$ topological space, (ii) $(X, \ast)$ is a group, and (iii) the maps $x \mapsto x^{-1}$ (from $X$ to $X$) and $(x, y) \mapsto x \ast y$ (from $X^2$ to $X$) are continuous.

Examples: $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$, $((0, \infty), \cdot)$, $(\mathbb{C} \setminus \{0\}, \cdot)$, $(S^1, \cdot)$, $GL(n, E)$, $SL(n, E)$ ($E = \mathbb{R}$ or $\mathbb{C}$), $O(n), U(n)$, $SO(n), SU(n)$, $(\mathbb{Q}, +)$, all normed spaces, any group with discrete topology, the group of isometries of $\mathbb{R}^n$, and ...

Exercise-22: Let $X$ be a compact metric space and $H(X, X) = \{f : X \to X : f$ is a homeomorphism$\}$. Then $H(X, X)$ is a topological group with respect to compact-open topology and composition. [Hint: The inverse map $f \mapsto f^{-1}$ is continuous since it maps $S(K, V)$ bijectively onto $S(X \setminus V, X \setminus K)$ (compactness of $X$ is used to say $X \setminus V$ is compact). Now, given $f, g \in H(X, X)$ and $S(K, V)$ with $f \circ g \in S(K, V)$, choose open $U$ such that $g(K) \subset U \subset \overline{U} \subset f^{-1}(V)$. Then the neighborhood $S(U, V) \times S(K, U)$ of $(f, g)$ is taken into $S(K, V)$ by the product map.]

Remark: With respect to the compact-open topology, $H(X, X)$ need not be a topological group if $X$ is only LCSC [J.J. Dijkstra, On homeomorphism groups and the compact-open topology, Amer. Math. Monthly, 112, (2005), 910-912]; is a topological group if $X$ is LCSC and locally connected [R.Arens, Topologies for homeomorphism groups, Amer. J. of Math., 68, (1946), 593-610].

Exercise-23: Let $(X, T)$ be a $T_1$ topological space and suppose $(X, \ast)$ is a group. Then $(X, T, \ast)$ is a topological group iff $(x, y) \mapsto x \ast y^{-1}$ is continuous.

Exercise-24: Show that the Cantor space $\{0,1\}^N$ with coordinatewise addition modulo 2 is a topological group. More generally, if $G_\alpha$’s are topological groups, then the direct product $\prod_\alpha G_\alpha$ with product topology and coordinatewise operation is a topological group, and the direct sum $\bigoplus_\alpha G_\alpha = \{(x_\alpha) \in \prod_\alpha G_\alpha : x_\alpha = e_\alpha$ except for finitely many $\alpha\}$ is a dense subgroup of $\prod_\alpha G_\alpha$.

[141] Let $X$ be a topological group. Then for each $a \in X$, the left translation $L_a : X \to X$, $L_a(x) = ax$, is a homeomorphism with $(L_a)^{-1} = L_a^{-1}$. Similarly, the right translation $R_a$ is also a homeomorphism. Consequently, every topological group is homogeneous. Then inverse map $x \mapsto x^{-1}$ is also a homeomorphism.

Non-examples: (i) $(\mathbb{R}, +)$ with cofinite topology is $T_1$ and homogeneous - translations are homeomorphisms. Also $x \mapsto -x$ is continuous. Now note that $\mathbb{R} \setminus \{1\}$ is an open neighborhood of 0, and that for any two nonempty open $U, V \subset \mathbb{R}$, there exist $a \in U, b \in V$ with $a + b = 1$. So the map $(x, y) \mapsto x + y$ is not continuous at $(0, 0)$. Another argument to say $(\mathbb{R}, +)$ with cofinite topology is not a topological group is to note that the space is not Hausdorff and then to use [142] below.
(ii) Consider \((\mathbb{R}, +)\) with the usual topology. Then \(\mathbb{R}/\mathbb{Q}\) is a quotient group. But the quotient topology on \(\mathbb{R}/\mathbb{Q}\) is indiscrete (\(\therefore\) if \(A \subseteq \mathbb{R}/\mathbb{Q}\) is nonempty open, then \(q^{-1}(A)\) is nonempty open in \(\mathbb{R}\) and \(q^{-1}(A) + \mathbb{Q} = q^{-1}(A)\) so that \(q^{-1}(A) = \mathbb{R}\) and hence not \(T_1\).

(iii) If \(\mathbb{R}/\mathbb{Q}\) is metrizable, (ii) every topological group is completely regular [p.70 of Hewitt and Ross].

(iv) Even if (iii) is compact and \(A\) and \(B\) are compact, then \(AB\) is not closed.

Proof. (i) Let \(\mathcal{T} = \{U \subseteq \mathbb{R} : U\) open in \(\mathbb{R}\) and there is \(M > 0\) such that \([M, \infty) \subseteq \mathbb{R} \setminus U\} \cup \{\mathbb{R}\). Then, in \((\mathbb{R}, \mathcal{T}, +), (x, y) \mapsto x + y\) is continuous but \(x \mapsto -x\) is not continuous.

The Hilbert cube \([0, 1]^\mathbb{N}\) is homogeneous (non-trivial to prove; see the book by van Mill - Infinite dimensional topology of function spaces). We note that any continuous self-map \(f\) on \([0, 1]^\mathbb{N}\) has a fixed point (\(\therefore\) define \(g_n : [0, 1]^n \rightarrow [0, 1]^n\) as \(g_n(x_1, \ldots, x_n) = \Pi_n(f(x_1, \ldots, x_n, 0, 0, 0, \ldots)),\) where \(\Pi_n : [0, 1]^n \rightarrow [0, 1]^n\) is the projection to the first \(n\) coordinates. Let \((a_1(n), \ldots, a_n(n))\) be a fixed point of \(g_n\), and let \(b(n) = (a_1(n), \ldots, a_n(n), 0, 0, 0, \ldots) \in [0, 1]^\mathbb{N}\). If \((b(n_k)) \rightarrow b\), then \(f(b) = b\). Now observe that a topological space \(X\) having fixed point property cannot be made into a topological group if \(|X| \geq 2\) (\(\therefore\) if \(a \in X \setminus \{e\}\), then \(L_a\) does not have fixed points). Thus the Hilbert cube does not admit the structure of a topological group.

[142] Let \(X\) be a topological group with identity \(e\).

(i) If \(V \subseteq X\) is a symmetric (i.e., \(V^{-1} = \{v^{-1} : v \in V\} = V\)) open set containing \(e\), then \(V \subseteq V^2 = \{vw : v, w \in V\}\).

(ii) If \(U \subseteq X\) is a neighborhood of \(e\), then there is a symmetric open set \(V \subseteq X\) containing \(e\) such that \(V^2 \subseteq U\).

(iii) Every topological group is regular.

Proof. (i) Let \(x \in V \setminus V\). Since \(xV\) is a neighborhood of \(x\), we have \(xV \cap V \neq \emptyset\). So there are \(v_1, v_2 \in V\) such that \(xv_1 = v_2\) or \(xv_2v_1^{-1} \in V^2\).

(ii) Since \((x, y) \mapsto xy\) is continuous at \(e\), there are neighborhoods \(V_1, V_2\) of \(e\) such that \(V_1V_2 \subseteq U\).

(iii) Because of homogeneity, it suffices to show the following: if \(U\) is a neighborhood of \(e\), then there is an open set \(V\) with \(e \in V \subseteq V \subseteq U\). But this follows from (i) and (ii).

Remark: From Urysohn’s metrization theorem, it follows that every second countable topological group is metrizable. In fact the following are true: (i) every topological group having a countable base at the identity is metrizable, (ii) every topological group is completely regular [p.70 of Hewitt and Ross].

[143] Let \(X\) be a topological group and \(A, B \subseteq X\).

(i) If \(A\) or \(B\) is open, then \(AB\) is open.

(ii) If \(A\) and \(B\) are compact, then \(AB\) is compact.

(iii) If \(A\) is compact and \(B\) is closed, then \(AB\) is closed.

(iv) Even if \(A, B\) are closed, \(AB\) need not be closed.

Proof. (i) \(AB = \bigcup_{a \in A} aB = \bigcup b \in BAb\).
(ii) $AB$ is the continuous image of the compact set $A \times B$ under the product map.

(iii) First assume $X$ is metrizable and let $x \in AB$. Then there are $(a_n)$ in $A$ and $(b_n)$ in $B$ such that $(a_n b_n) \to ab$. Since $A$ is compact, assume $(a_n b_n) \to a \in A$. Then, $(b_n) = (a_n^{-1} a_n b_n) \to a^{-1} x$ and therefore $a^{-1} x \in B$ since $B$ is closed. So $x = aa^{-1} x \in AB$. If $X$ is not metrizable, argue similarly using nets instead of sequences.

(iv) Let $X = (\mathbb{R}, +)$, $A = \mathbb{Z}$ and $B = \alpha \mathbb{Z}$, where $\alpha$ is irrational. Then $A, B$ are closed subgroups of $\mathbb{R}$, and $A + B \neq \mathbb{R}$ since $A + B$ is countable. To show $A + B$ is not closed, it now suffices to show that $A + B$ is dense in $\mathbb{R}$. Let $\epsilon = \inf\{ x \in A + B : x > 0 \}$. Enough to show $\epsilon = 0$ since $A + B$ is a subgroup. Let $k \in \mathbb{N}$ and let if possible $(A + B) \cap (0, 1/k) = \emptyset$; we will derive a contradiction. Since $(A + B) \cap (0, 1) \neq \emptyset$, there is a maximal $i \in \{1, 2, \ldots, k - 1\}$ such that $(A + B) \cap (0, i/k) = \emptyset$. Then there is $x \in (A + B) \cap (\frac{1}{k}, \frac{1}{k} + 1)$. Choose $n \in \mathbb{N}$ such that $nx < 1 < (n + 1)x$. Since $1 - nx > 0$ and $1 - nx > 1/k$ or $nx < 1 - 1/k$. Then, $1 < (n + 1)x < 1 + i/k$ and this means $(n + 1)x - 1 \in (A + B) \cap (0, i/k)$, a contradiction. \hfill $\square$

[144] Let $X$ be a topological group.

(i) If $U$ is a neighborhood of $\epsilon$ and $x \in X$, then there is a neighborhood $V$ of $\epsilon$ with $xV^{-1} \subset U$.

(ii) If $U$ is a neighborhood of $\epsilon$ and $K \subset X$ is compact, then there is a neighborhood $V$ of $\epsilon$ with $KV^{-1} \subset U$.

**Proof.** (i) $y \mapsto xy^{-1}$ is continuous at $\epsilon$.

(ii) Let $h : X^2 \to X$ be $h(x, y) = xy^{-1}$, which is continuous. For each $x \in K$, there exist neighborhoods $W(x)$ of $x$ and $V(x)$ of $\epsilon$ with $h(W(x) \times V(x)) = W(x)V(x)W(x)^{-1} \subset U$. Let $x_1, \ldots, x_n \in K$ be such that $K \subset \bigcup_{i=1}^n W(x_i)$ and let $V = \bigcap_{i=1}^n V(x_i)$. \hfill $\square$

**Exercise-25:** Let $X$ be a topological group, $A \subset X$ be compact, $B \subset X$ be closed, and $A \cap B = \emptyset$. Then there is a neighborhood $U$ of $\epsilon$ such that $AU \cap BU = \emptyset = UA \cap UB$. [Hint: Let $V$ be a neighborhood of $\epsilon$ disjoint with the closed sets $A^{-1}B$ and $AB^{-1}$, and let $U$ be a symmetric neighborhood of $\epsilon$ with $U^2 \subset V$.]

**Exercise-26:** Let $X$ be a topological group, $K \subset U \subset X$ where $K$ is compact and $U$ is open. Then there is a neighborhood $V$ of $\epsilon$ such that $KV \cup VK \subset U$. If $X$ is locally compact, $V$ can be chosen so that $KV \cup VK \subset U$ is compact. [Hint: For each $x \in K$, there is a neighborhood $V(x)$ of $\epsilon$ such that $xV(x) \subset U$ and $V(x)x \subset U$. If $K \subset \bigcup_{i=1}^n x_i V(x_i) \cap \bigcup_{i=1}^n V(x_i) x_i$, take $V = \bigcap_{i=1}^n V(x_i)$].

**Exercise-27:** Let $X$ be a topological group and $A, B \subset X$. Then, (i) $\overline{AB} \subset \overline{AB}$, (ii) $(\overline{A})^{-1} = \overline{(A^{-1})}$, (iii) $\overline{xy} = \overline{xy}$ for every $x, y \in X$. [Hint: $(x, y) \mapsto xy$ is continuous, and $z \mapsto z^{-1}$, $z \mapsto xyz$ are homeomorphisms.]

**Exercise-28:** Suppose $X$ is a group and $\{U_\alpha : \alpha \in I\}$ is a collection of subsets of $X$ such that (i) $e \in U_\alpha$ for every $\alpha$, (ii) for every $\alpha, \beta$, there is $\gamma$ such that $U_\gamma \subset U_\alpha \cap U_\beta$, (iii) for every $\alpha$, there is $\beta$ with $U_\beta^2 \subset U_\alpha$, (iv) for every $\alpha$, there is $\beta$ with $U_\beta^{-1} \subset U_\alpha$, (v) for every $\alpha \in I$ and for every
Theorem. Let \( X \) be a topological group, let \( Y \subset X \) be the commutator subgroup, i.e., \( Y \) is generated by \( \{aba^{-1}b^{-1} : a, b \in X\} \). We know from Algebra that \( Z, Y \) are normal subgroups. Show that: (i) \( Z \) is closed. (ii) \( Y \) is connected if \( X \) is connected. (iii) If \( X \) is connected and \( H \subset X \) is a totally disconnected normal subgroup, then \( H \subset Z \).  

**Proof.** (i) \( Z \) is closed. (ii) \( Y \) is connected. (iii) If \( X \) is connected and \( H \subset X \) is a totally disconnected normal subgroup, then \( H \subset Z \).
\( q^{-1}(V) = AY, q^{-1}(W) = BY \) are open in \( X \), we have that \( V, W \) are open in \( X/Y \). Check that 
\( VW^{-1} \subset U \) using the fact that \( q \) is a homomorphism.

(ii) If \( Z \subset X \) is open, then \( q^{-1}(q(Z)) = ZY \) is open in \( X \) so that \( q(Z) \) is open in \( X/Y \).

(iii) Compactness, connectedness and path connectedness are preserved by continuous maps. \( X/Y \) is locally compact: if \( U \subset X \) is an open neighborhood of \( e \) with compact closure, then \( q(\overline{U}) \) is compact and hence closed, and therefore \( q(U) \) is an open neighborhood of \( Y \) with compact closure \( \overline{q(U)} = q(\overline{U}) \). \( X/Y \) is second countable (totally disconnected): if \( B \) is an open neighborhood of \( e \) with compact closure \( B \) is a compactly generated. \( □ \)

Remark: In general, quotient maps of topological spaces are not open maps (\( \therefore \) let \( X = [0,2] \), collapse \([1,2] \) to the singleton \( \{1\} \) and note that the image of \((1,2) \) is \( \{1\} \)), and quotient maps of topological groups are not closed maps (\( \therefore \) under the map \((x, y) \mapsto x \) from \( \mathbb{R}^2 \) to \( \mathbb{R} \), consider the image of \( \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy = 1\} \)); an exception is:

Exercise-31: Let \( X \) be a topological group and \( Y \subset X \) be a compact normal subgroup. Then the quotient map \( q : X \rightarrow X/Y \) is a closed map. \([\text{Hint: If } A \subset X \text{ is closed, then } q^{-1}(q(A)) = AY \text{ is closed in } X \text{ by } [143].]\)

Exercise-32: Let \( X \) be a topological group and \( Y \subset X \) be a compact normal subgroup such that \( X/Y \) is compact. Then \( X \) is also compact. \([\text{Hint: Let } \{U_\alpha : \alpha \in I\} \text{ be an open cover for } X. \text{ If } x \in X, \text{ then there is finite } I(x) \subset I \text{ such that } xY \subset \bigcup_{\alpha \in I(x)} U_\alpha \text{ since } xY \text{ is compact. Choose open neighborhood } V(x) \text{ of } x \text{ such that } V(x)Y \subset \bigcup_{\alpha \in I(x)} U_\alpha. \text{ Since } q(V(x)) \text{ is open, there exist } x_1, \ldots, x_n \in X \text{ such that } X/Y = \bigcup_{i=1}^n q(V(x_i)). \text{ Then } \{U_\alpha : 1 \leq i \leq n, \alpha \in I(x_i)\} \text{ is a finite cover for } X.] \) (It is also true that if \( Y \) and \( X/Y \) are locally compact, so is \( X \) - p.39 of Hewitt and Ross.)

[147] Let \( X \) be a connected topological group. Then any neighborhood of \( e \) generates \( X \). In fact, if \( V \) is a symmetric neighborhood of \( e \), then \( X = \bigcup_{n=1}^\infty V^n \). In particular, if \( X \) is also locally compact, then \( X \) is compactly generated.

Proof. Given a neighborhood \( U \) of \( e \), chose a symmetric open neighborhood \( V \) of \( e \) with \( V \subset U \), and let \( Y = \bigcup_{n=1}^\infty V^n \). Clearly \( Y \) is a subgroup of \( X \). Since \( Y \) is open, it is also closed by [145]. Thus \( Y = X \) since \( X \) is connected. If \( X \) is locally compact, we may choose \( V \) with compact closure. Then, \( X = \bigcup_{n=1}^n \overline{V^n} \).

As topological groups \( \mathbb{R} \) and \( \{0,1\}^\mathbb{N} \) are two extreme cases; one is connected and the other is totally disconnected. The following result helps to reduce some arguments about topological groups to arguments about the two extreme cases: connected groups and totally disconnected groups.

[148] Let \( X \) be a topological group and let \( Y \subset X \) be the connected component of \( e \). Then \( Y \) is a closed normal subgroup of \( X \) and \( X/Y \) is totally disconnected.
Proof. Being a connected component in a topological space, \( Y \) is closed. Since \( YY^{-1} \) is the continuous image of \( Y \) under the map \((a,b) \mapsto ab^{-1}\), \( YY^{-1} \) is connected. And \( e \in Y \cap YY^{-1} \). Hence \( YY^{-1} \subset Y \) by the maximality of \( Y \). Similarly, being the continuous image of \( Y \) under the map \( a \mapsto xax^{-1}, xy^{-1} \) is connected and hence is contained in \( Y \) for each \( x \in X \). Thus \( Y \) is a normal subgroup.

By homogeneity, the connected components of \( X \) are precisely the cosets \( xY \). If \( A \subset X/Y \) is a set with \( |A| \geq 2 \), there is \( F \subset X \) with \( |F| \geq 2 \) such that \( q^{-1}(A) \) is the disjoint union \( q^{-1}(A) = \bigcup_{x \in F} xY \). Let \( U, V \subset X \) be open sets separating \( q^{-1}(A) \). Then for each \( x \in F \), we have either \( xY \subset U \) or \( xY \subset V \) since \( xY \) is connected. So \( F \) is the disjoint union of two nonempty sets \( F_1 = \{x \in F : xY \subset U\}, F_2 = \{x \in F : xY \subset V\} \). Then \( q(U), q(V) \) are open sets in \( X/Y \) intersecting \( A \) with \( A \subset q(U) \cup q(V) \) and we have \( A \cap q(U) = \{q(x) : x \in F_1\}, \ A \cap q(V) = \{q(x) : x \in F_2\} \) so that \( A \cap q(U) \cap q(V) = \emptyset \). Thus the sets \( q(U), q(V) \) form a separation for \( A \) and so \( A \) cannot be connected.

Exercise-33: Let \( X \) be a topological group and \( Y \subset X \) be a closed subgroup. If the topological spaces \( Y \) and \( X/Y \) are connected, then \( X \) is connected. \([\text{Hint: Similar to the proof of [148].}]\]

[149] Let \( X \) be a topological group and \( Y \subset X \) be a subgroup.

(i) If \( \{e\} \) is open in \( Y \), then \( Y \) is discrete.

(ii) If \( Y \) is locally compact (in particular, if \( Y \) is discrete), then \( Y \) is closed in \( X \) [note that this is not true for topological spaces].

(iii) If \( Y \) is closed in \( X \), the topological space \( X/Y \) is discrete iff \( Y \) is open.

Proof. (i) By homogeneity.

(ii) First suppose \( Y \) is discrete. Then there is a neighborhood \( U \) of \( e \) such that \( Y \cap U = \{e\} \). Choose a symmetric neighborhood \( V \) of \( e \) with \( V^2 \subset U \). Now, if \( Y \) is not closed, let \( x \in X \setminus Y \) be a limit point of \( Y \). Then there are distinct \( y_1, y_2 \in Y \) in the neighborhood \( xV \) of \( x \). Now, \( e \neq y_1^{-1}y_2 \in Y \) and \( y_1^{-1}y_2 = y_1^{-1}x^{-1}y_1 = (x^{-1}y_1)^{-1}x^{-1}y_2 \in V^{-1}V = V^2 \subset U \), a contradiction.

Now suppose \( Y \) is locally compact. There is a neighborhood \( U \) of \( e \) in \( X \) such that \( cl_Y[Y \cap U] \) is compact. Let \( W \subset X \) be open such that \( e \in W \subset \overline{W} \subset U \). Then \( Y \cap \overline{W} \) is compact, being closed in \( cl_Y[Y \cap U] \). Therefore, \( Y \cap \overline{W} \) is closed in \( X \). Let \( V \) be a symmetric neighborhood of \( e \) with \( V^2 \subset W \). If \( x \in \overline{Y} \) is a limit point of \( Y \), then there is \( y \in Y \cap xV \). We claim that \( y^{-1}x \) is a limit point of \( Y \). If the claim is true, then \( y^{-1}x \in Y \cap \overline{W} \) or \( x \in yY = Y \). To prove the claim consider a neighborhood \( O \) of \( y^{-1}x \). Then \( yO \cap xV \) is a neighborhood of \( x \) and so there is \( z \in Y \cap yO \cap xV \). Then \( y^{-1}z \in Y \cap O \). Since \( y, z \in xV \), we also have \( y^{-1}z = (x^{-1}y)^{-1}x^{-1}z \in V^{-1}V = V^2 \subset W \). Hence \( y^{-1}z \in O \cap (Y \cap \overline{W}) \).

(iii) \( X/Y \) is discrete iff \( \{Y\} \) is open in \( X/Y \) iff \( Y \) is open in \( X \).
Corollary: Let $X$ be a connected topological group and let $Y$ be a locally compact subgroup having finite index in $X$. Then $Y = X$.

Proof. $Y$ is closed by [149] and then open by [145]. \qed

[150] Let $X$ be a topological group and let $U$ be a compact open neighborhood of $e$. Then there is a compact open subgroup $Y$ of $X$ contained in $U$. Consequently, we have the following:

(i) If $X$ is locally compact and totally disconnected, then there is a base at $e$ consisting of compact open subgroups.

(ii) If $X$ is compact and totally disconnected, then there is a base at $e$ consisting of compact open normal subgroups.

Proof. Since the compact $U$ is contained in the open $U$, by Exercise-26 there is an open neighborhood $W$ of $e$ such that $ UW \subset U$. We may assume that $W$ is symmetric and $W \subset U$. Then, $Y = \bigcup_{n=1}^{\infty} W^n$ is an open and hence closed subgroup of $X$. Since $W^2 \subset UW \subset U$, by induction $W^{n+1} = W^nW \subset UW \subset U$ and therefore $Y \subset U$.

(i) Fix a base $B$ at $e$ consisting of compact open subsets, and for each $U \in B$ choose a compact open subgroup $Y \subset U$.

(ii) Let $Y$ be a base at $e$ consisting of compact open subgroups. If $Y \in Y$, let $Z = \bigcap_{x \in X} xY x^{-1}$. Then $Z$ is a compact normal subgroup. Since $X$ is compact and $Y$ is a neighborhood of $e$, by [144] there is a neighborhood $W$ of $e$ such that $x^{-1}Wx \subset Y$ for every $x \in X$, or $W \subset \bigcap_{x \in X} xY x^{-1} = Z$. So $\text{int}[Z] \neq \emptyset$ and therefore $Z$ is open. Such $Z$’s form a base at $e$. \qed

Remarks: (i) A deep theorem (answer to Hilbert’s fifth problem) says that a topological group admits the structure of a Lie group iff it is locally compact and does not have ‘small subgroups’ (i.e., there is no base at $e$ consisting of subgroups). In this sense, locally compact totally disconnected groups are and Lie groups are at two extremes. (ii) Now we have yet another proof for the fact that $\mathbb{Q}$ is not locally compact: $(\mathbb{Q}, +)$ is totally disconnected, but does not have any compact open subgroup in $(-\epsilon, \epsilon)$ if $\epsilon > 0$.

Example: $X = \{0, 1\}^\mathbb{N}$ is a compact totally disconnected abelian group. A base consisting of compact open (normal) subgroups at $e = (0, 0, \ldots)$ is $\{Y_k : k \in \mathbb{N}\}$, where $Y_k = \{x \in X : x_n = 0$ for $1 \leq n \leq k\}$. More generally, if $X = \prod_{\alpha \in I} X_\alpha$, where $X_\alpha$’s are finite discrete groups, then a base consisting of compact open normal subgroups at $e \in X$ is $\{Y_F : F \subset I \text{ is finite}\}$, where $Y_F = \{x \in X : x_\alpha = e_\alpha \text{ for every } \alpha \in F\}$.

Exercise-34: Let $X$ be a locally compact group and $Y \subset X$ be the connected component of $e$. Then $Y$ is the intersection of all open subgroups of $X$. [Hint: If $Z \subset X$ is an open subgroup, then $Z$ is clopen and hence $Y \subset Z$. Now, $X/Y$ is locally compact and totally disconnected so that if $x \in X \setminus Y$, then there is a compact open subgroup $H$ of $X/Y$ such that $q(x) \notin H$ by [150]. Then, $q^{-1}(H)$ is an open subgroup of $X$ not containing $x$.]
Exercise-35: Let $X$ be a locally compact group. Then $X$ is a normal topological space. [Hint: Choose a symmetric open neighborhood $W$ of $e$ with compact closure. Then $Y = \bigcup_{n=1}^{\infty} W^n$ is a clopen subgroup of $X$. If $K_1 = \overline{W}$ and $K_{n+1} = \overline{W^{n+1}} \setminus W^n$, then $K_n$’s are compact, $Y = \bigcup_{n=1}^{\infty} K_n$ and $K_{n+j} \cap K_n = \emptyset$ if $j \geq 2$. If $A, B \subset Y$ are relatively closed, disjoint sets, choose open $U_n, V_n \subset X$ separating $K_n \cap A$ and $K_n \cap B$. Then the open sets $U = \bigcup_{n=1}^{\infty} U_n, V = \bigcup_{n=1}^{\infty} V_n$ separate $A$ and $B$, and thus $Y$ is a normal topological space. Now write $X = \bigcup_{\alpha} x_{\alpha} Y$, a disjoint union of cosets, and let $A, B \subset X$ be disjoint closed sets. Separate $A \cap x_{\alpha} Y$ and $B \cap x_{\alpha} Y$ by open sets $U_{\alpha}, V_{\alpha} \subset x_{\alpha} Y$. Then $U = \bigcup_{\alpha} U_{\alpha}$ and $V = \bigcup_{\alpha} V_{\alpha}$ are open sets separating $A, B$.] (Related fact: $\sigma$-compact Hausdorff topological space is normal.)

Continuous homomorphisms between topological groups: If $X, Y$ are topological groups and $f : X \to Y$ is a bijection such that $f, f^{-1}$ are continuous group homomorphisms, then we say $f$ is a topological isomorphism. An example is $f : (\mathbb{R}, +) \to ((0, \infty), \cdot), f(x) = e^x$. A non-example: $det : GL(n, \mathbb{R}) \to (\mathbb{R} \setminus \{0\}, \cdot)$ is a continuous surjective homomorphism, but it is not 1-1 if $n \geq 2$.

Considering the identity map from $(\mathbb{R}, \text{discrete topology}, +)$ to $(\mathbb{R}, \text{usual topology}, +)$, we see that the analogue of the First Isomorphism Theorem of groups fails in the category of topological groups in the absence of extra assumptions.

[151] [First Isomorphism Theorem] Let $X, Y$ be topological groups and let $f : X \to Y$ be a continuous open surjective homomorphism. Then $Z := \ker(f)$ is a closed normal subgroup of $X$ and $Y$ is topologically isomorphic to $X/Z$ via the map $y \mapsto f^{-1}(y)$.

Proof. From Algebra we know that $Z$ is a normal subgroup, and $Z$ is closed since $f$ is continuous. Note that if $f(x) = y$, then $f^{-1}(y) = xZ$. If $g : Y \to X/Z$ is $g(y) = f^{-1}(y)$, then $g$ is a group isomorphism. For $V \subset Y$, we have $g(V) = g(f^{-1}(V))$, where $g : X \to X/Z$ is the quotient map. Since both $g$ and $f$ are open and continuous, so is $g$.

[152] [Open Mapping Theorem] Let $X$ be a locally compact $\sigma$-compact group, $Y$ be a locally compact group and $f : X \to Y$ be a continuous surjective group homomorphism. Then $f$ is an open map.

Proof. By homogeneity, it suffices to show the following: if $U \subset X$ is a neighborhood of $e_X$, then there is a neighborhood $W \subset Y$ of $e_Y$ such that $W \subset f(U)$.

Let $V \subset X$ be a symmetric neighborhood of $e_X$ such that $V$ is compact and $V^2 \subset U$. Write $X = \bigcup_{j=1}^{\infty} K_j$, a countable union of compact sets. Since each $K_j$ can be covered by finitely many $xV$’s, there exist $x_1, x_2, \ldots \in X$ such that $X = \bigcup_{n=1}^{\infty} x_n V = \bigcup_{n=1}^{\infty} x_n V$. Then, $Y = f(X) = \bigcup_{n=1}^{\infty} f(x_n) f(V)$ since $f$ is a surjective homomorphism. Now, $f(V)$ is compact and hence closed. Since $Y$ is locally compact, $Y$ has Baire property by [104]. We conclude that $Z := \text{int}[f(V)] \neq \emptyset$. Let $z \in Z$ and $W = z^{-1}Z$. Then $W$ is a neighborhood of $e_Y$ and $W \subset Z^{-1}Z \subset f(V)^{-1} f(V) = f((V)^{-1} V) = f(V^2) \subset f(U)$.
[152'] [Corollary] Let $X$ be a locally compact $\sigma$-compact group, $Y$ be a locally compact group and let $f : X \to Y$ be a continuous surjective group homomorphism. Then $f$ is an open map and $X/\ker(f)$ is topologically isomorphic to $Y$.

**Examples:** (i) Let $f : \mathbb{R} \to S^1$ be $f(x) = e^{2\pi i x}$. Then $f$ is a continuous surjective group homomorphism with $\ker(f) = \mathbb{Z}$. Hence $\mathbb{R}/\mathbb{Z}$ is topologically isomorphic to $S^1$ by [152']. (ii) If $n \geq m$, any surjective linear map $f : \mathbb{C}^n \to \mathbb{C}^m$ is an open map. (iii) For $E = \mathbb{R}$ or $\mathbb{C}$, $\det : GL(n, E) \to (E \setminus \{0\}, \cdot)$ is an open map and $GL(n, E)/SL(n, E)$ is topologically isomorphic to $(E \setminus \{0\}, \cdot)$ by [152'].

**Exercise-36:** Let $X, Y$ be compact groups with at least two elements. If $X$ is totally disconnected and $Y$ is connected, then there does not exist any continuous surjective group homomorphism $f : X \to Y$ (contrast with the topological result [128]). [*Hint:* Let $y \in Y \setminus \{e_Y\}$ and let $G \subset X$ be a compact open subgroup. Then $f(G)$ is a compact open subgroup of $Y$ and hence $f(G) = Y$. So $y \in f(G)$. Since $X$ has a base at $e$ consisting of such $G$'s, we get by continuity that $f(e_X) = y \neq e_Y$, a contradiction.]

**Exercise-37:** [Second Isomorphism Theorem] Let $X, Y$ be topological groups, $f : X \to Y$ be an open continuous surjective group homomorphism, $N = \ker(f)$, $Z \subset Y$ be a closed normal subgroup and let $T = f^{-1}(Z)$. Then $X/T$, $Y/Z$ and $(X/N)/(T/N)$ are topologically isomorphic. [*Hint:* Verify first that $T$ in $X$ and $(T/N)$ in $(X/N)$ are closed normal subgroups. If $q : Y \to Y/Z$ is the quotient map, then $q \circ f : X \to Y/Z$ is an open continuous surjective homomorphism with kernel $T$ and hence $X/T \cong Y/Z$. Now, $y \mapsto f^{-1}(y)$ is a topological isomorphism from $Y$ to $X/N$ and the image of $Z$ under this map is $T/N$. So $Y/Z \cong (X/N)/(T/N)$].

[153] Let $X, Y$ be topological groups and $f : X \to Y$ be a group homomorphism that is continuous at some point of $X$. Then,

(i) $f$ is continuous.

(ii) If $X, Y$ admit left-invariant (or, right-invariant) metrics, then $f$ is uniformly continuous.

**Proof.** (i) By homogeneity. (ii) Let $d, d'$ be left-invariant metrics on $X, Y$ respectively. Given $\epsilon > 0$ choose $\delta > 0$ for the point $e_X$. Then for $a, b \in X$ with $d(a, b) < \delta$, we have $d(e_X, a^{-1}b) < \delta$ so that $\epsilon > d'(e_Y, f(a^{-1}b)) = d'(e_Y, f(a)^{-1}f(b)) = d'(f(a), f(b))$. \(\square\)

**Exercise-38:** Let $X$ be a locally compact group and $f : X \to \mathbb{R}$ be a continuous function with compact support. Then for every $\epsilon > 0$, there is a neighborhood $U$ of $e$ such that $|f(x) - f(y)| < \epsilon$ for every $x, y \in X$ with $x^{-1}y \in U$. [*Hint:* For each $a \in K := \text{supp}(f)$, there is a neighborhood $W(a)$ of $e$ such that $|f(a) - f(b)| < \epsilon/2$ for every $b \in aW(a)$. Let $U(a)$ be a symmetric neighborhood of $e$ with $U(a)^2 \subset W(a)$. Choose $a_1, \ldots, a_n \in K$ such that $K \subset \bigcup_{i=1}^{n} a_iU(a_i)$ and let $U = \bigcap_{i=1}^{n} U(a_i)$. Suppose $x^{-1}y \in U$. If $x \in K$, then $x \in a_iU(a_i)$ for some $i$, and then $y = x(x^{-1}y) \in a_iU(a_i)U \subset$
an admissible left-invariant metric on $a_iU(a_i)^2 \subset a_iW(a_i)$. Therefore $|f(x) - f(y)| \leq |f(x) - f(a_i)| + |f(a_i) - f(y)| < \epsilon/2 + \epsilon/2$. If $x, y \notin K$, then $f(x) = 0 = f(y)$.

**Example:** $d(x, y) = |\log x - \log y|$ is an invariant metric on $((0, \infty), \cdot)$.

**Example:** Let $(X, d)$ be a compact metric space and consider $H(X, X)$ with the supremum metric $\rho$. Easily seen that $\rho$ is right-invariant. But it is not left-invariant in general. Let $X = [0, 1]$ and $f, g \in H(X, X)$ be $f(x) = x$, $g(x) = x^2$. Then $\rho(f, g) = |f(1/2) - g(1/2)| = 1/4$. Let $h \in H(X, X)$ be such that $h(f(1/2)) - h(g(1/2)) = h(1/2) - h(1/4) > 1/4$. Then $\rho(h \circ f, h \circ g) > 1/4$.

**Example:** Let $n \geq 2$ and consider $GL(n, E)$, $E = \mathbb{R}$ or $\mathbb{C}$. If $A \in GL(n, E)$, let $\|A\|$ be either the Euclidean norm or the operator norm. The metric induced by this norm is neither left-invariant nor right-invariant: if $A = [a_{ij}] \in GL(n, E)$ is the diagonal matrix with $a_{11} = 2$ and $a_{ii} = 1$ for $i \geq 2$, then $\|I - A\| = 1 \neq 1/2 = \|A^{-1} - I\|$. However, $d(A, B) = \log (1 + \|I - A^{-1}B\| + \|I - B^{-1}A\|)$ is an admissible left-invariant metric on $GL(n, E)$.

**Observation:** If $d$ is an invariant metric (i.e., both left-invariant and right-invariant) on a topological group $X$ and if $(x_n), (y_n)$ are sequences in $X$ with $(x_n, y_n) \rightarrow e$, then $(y_n, x_n) \rightarrow e$ and $d(e, y_n, x_n) = d(y_n^{-1}, x_n) = d(y_n^{-1}x_n^{-1}, e) = d((x_n, y_n)^{-1}, e)$.

**Example:** For $n \geq 2$, $GL(n, E)$ and $SL(n, E)$ do not admit invariant metrics. We argue as follows. For reals $x, y$ with $x \neq 0$, let $A(x, y) \in SL(n, E) \subset GL(n, E)$ be the matrix with $a_{11} = x$, $a_{12} = y$, $a_{22} = 1/x, a_{ii} = 1$ for $i > 2$ and $a_{ij} = 0$ for all other $ij$. Note that $A(x, y)A(s, t) = A(xs, xt + y/s)$. Hence $A(1/m, 1/m)A(m, 1/m) = A(1, 2/m^2) \rightarrow I$ as $m \rightarrow \infty$, but $A(m, 1/m)A(1/m, 1/m) = A(1, 2) \neq I$. The argument is completed by the Observation given above.

**Example:** Every discrete topological group trivially admits an invariant metric, namely the discrete metric. Therefore, there are non-abelian topological groups admitting an invariant metric. More examples of such groups follow from [154] below.

**Remarks:** (i) It is a fact that if a topological group is metrizable, the it admits a left-invariant (right-invariant) metric [p.70 of Hewitt and Ross]. (ii) As mentioned earlier, a topological group is metrizable iff it has a countable base at the identity. For compact groups, this can be improved to:

[154] For a compact topological group $X$, the following are equivalent:

(i) $X$ has a countable base at $e$.

(ii) There is a continuous map $f : X \rightarrow \mathbb{R}$ such that $f(x) = 0$ iff $x = e$.

(iii) $X$ admits an invariant metric.

(iv) $X$ is metrizable.

In particular, compact first countable groups admit an invariant metric.

**Proof.** $(i) \Rightarrow (ii)$: Let $\{U_n : n \in \mathbb{N}\}$ be a countable base at $e$. Since $X$ is compact Hausdorff, $X$ is normal and hence completely regular. So for each $n$, there is continuous $f_n : X \rightarrow [0, 1]$
such that \( f_n(e) = 0 \) and \( f_n \equiv 1 \) on \( X \setminus U_n \). If \( f : X \to \mathbb{R} \) is \( f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x) \), then \( f \) is continuous since the series is uniformly convergent by Weierstrass M-test. Also \( f(x) = 0 \) iff \( x = e \) since \( \bigcap_{n=1}^{\infty} U_n = \{ e \} \).

(ii) \( \Rightarrow \) (iii): Define \( d(x, y) = \sup \{|f(axb) - f(ayb)| : a, b \in X\} \). Then \( d \) is an invariant metric on the group \( X \). Let \( T \) be the original topology on \( X \) and let \( T' \) be the topology induced by \( d \). First we show \( T' \subset T \). Given \( \epsilon > 0 \), by Exercise-38 choose a neighborhood \( U \) of \( e \) such that \(|f(x) - f(y)| < \epsilon/2\) for every \( x, y \in X \) with \( x^{-1}y \in U \). Then by [144], there is a neighborhood \( W \) of \( e \) such that \( xWx^{-1} \subset U \) for every \( x \in X \). If \( z \in W \) and \( a, b \in X \), then \((aeb)^{-1}(azb) = b^{-1}zb \in b^{-1}Wb \subset U \) so that \( |f(aeb) - f(azb)| < \epsilon/2 \) and therefore \( d(e, z) \leq \epsilon/2 < \epsilon \). That is, \( e \in W \subset B_{d}(e, \epsilon) \) and thus \( T' \subset T \). Hence \( Id : (X, T) \to (X, T') \) is a continuous bijection from a compact space to a Hausdorff space, and therefore is a homeomorphism. That is, \( T' = T \).

The implications (iii) \( \Rightarrow \) (iv) and (iv) \( \Rightarrow \) (i) are clear.

Example: By the above result, we conclude that if \( G = O(n), U(n), SO(n) \) or \( SU(n) \), then \( G \) admits an invariant metric. In fact, an invariant metric is \( d(A, B) = \sup \{|f(PAQ) - f(PBQ)| : P, Q \in G\} \), where \( f(A) = ||I - A|| \).

Remarks: (i) A corollary of [154] is that if \( X \) is a compact first countable group, then \( card[X] \leq card[\mathbb{R}] \). (ii) \( f : GL(n, E) \to \mathbb{R} \) given by \( f(A) = ||I - A|| \) is a continuous map satisfying \( f(A) = 0 \) iff \( A = I \), but we know that there is no invariant metric on \( GL(n, E) \) if \( n \geq 2 \). Thus we cannot replace compactness with local compactness in [154]. However, all locally compact groups are normal by Exercise-35.

3. Characters

While studying spaces it is important to know whether distinct points can be separated using a relevant class of functions. Examples: (i) If \( X \) is a normal topological space and \( x, y \in X \) are distinct points, then there is continuous \( f : X \to [0, 1] \) such that \( f(x) = 0 \neq 1 = f(y) \). (ii) [Corollary of Hahn-Banach Theorem] If \( X \) is a normed space and \( x, y \in X \) are distinct points, then there is a bounded linear functional \( f \) on \( X \) such that \( ||f|| = 1 \) and \( f(x) - f(y) = ||x - y|| \).

The ‘relevant’ class of functions for a topological group is the class of ‘characters’ defined as follows. If \( X \) is a topological group, a character of \( X \) is a continuous group homomorphism \( f : X \to (S^1, \cdot) \).

Question: If \( X \) is a topological group and \( x, y \in X \) are distinct points, does there exist a character \( f \) of \( X \) such that \( f(x) \neq f(y) \)? Equivalently, if \( x \in X \setminus \{e\} \), does there exist a character \( f \) of \( X \) such that \( f(x) \neq 1 \)?

Remark: Suppose a topological group \( X \) is not abelian. Let \( x, y \in X \) be such that \( xy \neq yx \), i.e, \( xyx^{-1}y^{-1} \neq e \). Since \( S^1 \) is abelian, for any character \( f \) of \( X \) we have \( f(xyx^{-1}y^{-1}) = f(x)f(y)f(x)^{-1}f(y)^{-1} = 1 \). Therefore, being abelian is a necessary condition for a topological
group to have sufficiently many characters to separate points. Later we will see that locally compact abelian groups have enough characters (to separate points). Because of this, locally compact abelian groups have a nice theory.

Exercise-39: If $H$ is a closed subgroup of $(\mathbb{R}, +)$, then $H = \{0\}, \mathbb{R}$ or $\alpha\mathbb{Z}$ for some $\alpha > 0$. [Hint: Suppose $H \neq \{0\}$. If $H \cap (0, \epsilon) \neq \emptyset$ for every $\epsilon > 0$, then $H$ will be dense in $\mathbb{R}$. So if $H \neq \mathbb{R}$, then $\alpha := \inf \{\epsilon > 0 : H \cap (0, \epsilon) = \emptyset\} > 0$. Show that $H = \alpha\mathbb{Z}$.]

Exercise-40: If $H$ is a closed subgroup of $(\mathbb{S}^1, \cdot)$, then $H = \mathbb{S}^1$ or \{\textit{n}th roots of unity\} for some $n \in \mathbb{N}$. [Hint: If $g : \mathbb{R} \to \mathbb{S}^1$ is $g(x) = e^{2\pi ix}$, then $g^{-1}(H)$ is a closed subgroup of $(\mathbb{R}, +)$. Also $g(\alpha\mathbb{Z}) = g(\mathbb{Z} + \alpha\mathbb{Z})$ for every $\alpha \in \mathbb{R}$ and $\mathbb{Z} + \alpha\mathbb{Z}$ is dense in $\mathbb{R}$ if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Hence if $g^{-1}(H) \neq \{0\}, \mathbb{R}$, then $g^{-1}(H) = \alpha\mathbb{Z}$ for some $\alpha \in \mathbb{Q}$.]

[155] (i) $f$ is a character of $(\mathbb{Z}, +)$ iff there is $a \in \mathbb{S}^1$ such that $f(n) = a^n$ for every $n \in \mathbb{Z}$.
(ii) $f$ is a character of $(\mathbb{R}, +)$ iff there is $y \in \mathbb{R}$ such that $f(x) = e^{2\pi ixy}$ for every $x \in \mathbb{R}$.
(iii) $f$ is a character of $(\mathbb{S}^1, \cdot)$ iff there is $n \in \mathbb{Z}$ such that $f(a) = a^n$ for every $a \in \mathbb{S}^1$.

Proof. The implication “$\Leftarrow$” is easy to verify in all the cases. We prove “$\Rightarrow$”.

(i) Let $a = f(1)$.

(ii) Let $H = \ker(f)$, which is a closed subgroup of $\mathbb{R}$. Hence $H = \{0\}, \mathbb{R}$ or $\alpha\mathbb{Z}$ for some $\alpha > 0$. If $H = \{0\}$, $f$ becomes a group isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{S}^1, \cdot)$, a contradiction since $\mathbb{S}^1$ has non-trivial elements of finite order. If $H = \mathbb{R}$, take $y = 0$. Now suppose that $H = \alpha\mathbb{Z}$, $\alpha > 0$. Then $f$ is 1-1 on $(0, \alpha)$ and $f([0, \alpha])$ must be connected. So $f$ maps $[0, \alpha)$ continuously and bijectively onto $\mathbb{S}^1$. Moreover, $f$ must take $\{k\alpha/n : 0 \leq k \leq n - 1\}$ onto the $n$th roots of unity since $f$ is a homomorphism. First assume that $f$ is orientation-preserving. Then $f(k\alpha/n) = e^{2\pi ik/n} = e^{2\pi i(k\alpha/n)(1/\alpha)}$ for $n \in \mathbb{N}$ and $0 \leq k < n - 1$. Since $f(x + ma) = f(x)$ for $m \in \mathbb{Z}$, it follows that $f(x) = e^{2\pi i(x/\alpha)}$ for every $x \in \mathbb{Q}$, and then for every $x \in \mathbb{R}$ by the continuity of $f$. Thus we take $y = 1/\alpha$. If $f$ is orientation-reversing, by a similar argument, $y = -1/\alpha$.

(iii) First argument: if $H = \ker(f)$, then $H$ is a closed subgroup of $\mathbb{S}^1$; now make use of Exercise-40. If $H = \{1\}$, then $f$ is a continuous bijection of $\mathbb{S}^1$ and hence is a homeomorphism. Since $f$ is a group isomorphism, $f$ must act as a permutation on \{\textit{m}th roots of unity\} for each $m \in \mathbb{N}$. Conclude that $f = \pm \text{Id}$. So take $n = \pm 1$. If $H = \mathbb{S}^1$, take $n = 0$. If $H = \{\text{m}th roots of unity\}$, then $f$ maps the arc $[1, e^{2\pi i/m})$ bijectively and continuously onto $\mathbb{S}^1$. Depending upon $f$ preserves or reverses orientation, try to conclude that $n = m$ or $n = -m$.

Second argument: If $g : \mathbb{R} \to \mathbb{S}^1$ is $g(x) = e^{2\pi ix}$, then $f \circ g$ is a character of $\mathbb{R}$ and hence by (ii), there is $y \in \mathbb{R}$ such that $f(g(x)) = f(e^{2\pi ix}) = e^{2\pi ixy}$ for every $x \in \mathbb{R}$. Since $g(x + 1) = g(x)$, we have $e^{2\pi i(x+1)y} = e^{2\pi ixy}$ for every $x \in \mathbb{R}$ or $e^{2\pi iy} = 1$, which gives $y \in \mathbb{Z}$. That is, $f(a) = a^y$ for $a \in \mathbb{S}^1$. Take $n = y$. □

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